

Ergodic theory of amenable semigroup actions

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Abstract. In this paper, among other things, we state and prove the mean ergodic theorem for amenable semigroup algebras.

Keywords. Asymptotically invariance property; Banach algebras; locally compact semigroup; mean ergodic theorem; topologically left invariant mean; topologically right invariant mean; weakly almost periodic.

1. Introduction

Actions of amenable semigroups in connection with fixed point conditions were considered in [8]. The classical mean ergodic and pointwise ergodic theorems are concerned with the convergence of the sequence of Cesaro means [14]. The mean ergodic theorem for amenable locally compact groups was proved by Greenleaf [9,10] (for more on mean ergodic theorem on locally compact group, the reader is referred to [10] and [14]). For the pointwise ergodic theorem, refer [14].

Our primary concern in this paper is to prove a theorem for an amenable locally compact semigroup S . Other aspects of the relationship between amenability and ergodic theory are mentioned in Corollaries 1, 2. A detailed historical account on the notion of ergodicity is given in [13].

Let S be a locally compact Hausdorff semitopological semigroup. Let $M(S)$ be the Banach algebra of all bounded, regular Borel measures on S with total variation norm and convolution as multiplication, and let $M_0(S)$ be the semigroup of all probability measures in $M(S)$. Let $M(S)^*$ be the continuous dual of $M(S)$, and 1 the linear functional in $M(S)^*$ such that $\langle 1, \mu \rangle = \mu(S)$ for all μ in $M(S)$.

Recall that on $M(S)^{**}$ we define the first Arens product by

$$\langle f\mu, \nu \rangle = \langle f, \mu * \nu \rangle, \quad \langle Ff, \mu \rangle = \langle F, f\mu \rangle, \quad \langle GF, f \rangle = \langle G, Ff \rangle,$$

where $\mu, \nu \in M(S)$, $f \in M(S)^*$ and $F, G \in M(S)^{**}$. A linear functional $M \in M(S)^{**}$ is called a *mean* if $\langle M, f \rangle \geq 0$ whenever $f \geq 0$ and if $\langle M, 1 \rangle = 1$. A mean M is *topological left (right) invariant* if $\langle M, f\mu \rangle = \langle M, f \rangle$ ($\langle M, \mu f \rangle = \langle M, f \rangle$) for any $\mu \in M_0(S)$ and $f \in M(S)^*$. We know that topologically left invariant means on $M(S)^*$ have been studied by Riazi and Wong [16] and by Wong [18,19,20]. We shall follow [8] and [16] for definitions and terminologies not explained here.

2. Main results

Throughout the paper, S is a locally compact Hausdorff semitopological semigroup. A semigroup S is said to *act* on a set X (from the right) if there exists a mapping

$$(\xi, s) \mapsto \xi s, X \times S \rightarrow X$$

such that $\xi(st) = (\xi s)t$ whenever $s, t \in S$ and $\xi \in X$; S is a *transformation semigroup* on X . We say that the complex Banach space X with a mapping $(\xi, s) \mapsto \xi s$ constitutes a *right Banach S -module* if the mapping has the following properties:

- (a) for each fixed $s \in S$, the mapping $\xi \mapsto \xi s$ is linear on X ;
- (b) there exists a positive constant k such that $\|\xi s\| \leq k\|\xi\|$ ($\xi \in X, s \in S$) and, for every $\xi \in X$, the mapping $s \mapsto \xi s$ is continuous.

We define similarly a *left dual S -module structure* on X^* by putting $\langle sf, \xi \rangle = \langle f, \xi s \rangle$ whenever $s \in S, \xi \in X$ and $f \in X^*$. For every $f \in X^*, s \mapsto sf$ is a continuous mapping of S into X^* when X^* is equipped with the weak*-topology.

Let X be a right Banach S -module. We may give X a right Banach $M(S)$ -module structure via a vector-valued integral. We put

$$\xi \mu = \int \xi s d\mu(s),$$

for $\mu \in M(S), \xi \in X$. Notice also that if $s \in S$ and $\xi \in X$, then $\xi \delta_s = \xi s$. Finally, for every $\mu \in M(S), \xi \in X$ and $f \in X^*$, we define $\langle \mu, f\xi \rangle = \langle f, \xi \mu \rangle$.

PROPOSITION 1.

Let $\xi \in X$ be such that $C_\xi = \overline{\text{co}}\{\xi \delta_s; s \in S\}$ is weakly compact. Then $C_\xi = \overline{\{\xi \mu; \mu \in M_0(S)\}}$.

Proof. It is clear that $C_\xi \subseteq \overline{\{\xi \mu; \mu \in M_0(S)\}}$. We will see that this inclusion is actually an equality. To reach a contradiction, assume that some $\xi \mu \in \{\xi \mu; \mu \in M_0(S)\}$ is not in C_ξ . Theorem 3.4(b) of [17] shows that there exist $f \in X^*$ and $\gamma \in \mathbb{R}$ such that, for every $s \in S$,

$$\text{Re}\langle f, \xi s \rangle < \gamma < \text{Re}\langle f, \xi \mu \rangle.$$

Therefore

$$\text{Re}\langle f, \xi \mu \rangle = \int \text{Re}\langle f, \xi s \rangle d\mu(s) < \text{Re}\langle f, \xi \mu \rangle,$$

which is contradiction. \square

Let Ω be a subset of $M(S)$. We say that a net (μ_α) in $M_0(S)$ is *asymptotically Ω -invariant* if $\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$ whenever $\mu \in \Omega$. Asymptotical invariance property was introduced by Day [4]. Wong [18] demonstrated that $M(S)^*$ has a topologically left invariant mean if and only if there exists a net (μ_α) in $M_0(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$ whenever $\mu \in M_0(S)$. If S is a semi-foundation semigroup, the following conditions [7] are equivalent:

- (a) $M(S)^*$ has a topologically left invariant mean;

- (b) there is a net (μ_α) in $M_0(S)$ such that, for every compact subset K of S , $\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$ uniformly over all μ in $M_0(S)$ which are supported on K .

Let $M(S)^*$ have a topologically left invariant mean, and let (μ_α) be a net in $M_0(S)$. It is easy to see that (μ_α) is asymptotically $M_0(S)$ -invariant if and only if

$$\lim_{\alpha} \|\mu * \mu_\alpha\| = |\mu(S)|$$

whenever $\mu \in M(S)$.

Let X be a right Banach S -module. Let (μ_α) be an asymptotically $M_0(S)$ -invariant in $M_0(S)$. For each α , we define $E_\alpha[\xi] \in X$ by

$$E_\alpha[\xi] = \int \xi s d\mu_\alpha(s), \quad \xi \in X.$$

We now concentrate on the mean ergodic theorem for a locally compact semigroup. The following result can be referred to as ‘the mean ergodic theorem’ for amenable locally compact semigroups (see [10,14,15], for details).

Theorem 1. *Let $M(S)^*$ have a topologically right invariant mean. Suppose (μ_α) is an asymptotically $M_0(S)$ -invariant in $M_0(S)$.*

- (1) *If $\xi \in X$ is such that $C_\xi = \overline{\text{co}}\{\xi \delta_s; s \in S\}$ is weakly compact, then $(E_\alpha[\xi])$ converges to a fixed point of C_ξ that is the unique fixed point in C_ξ , and is therefore independent of the choice of (μ_α) .*
- (2) *Suppose that C_ξ is weakly compact for each $\xi \in X$. Let $Y = \{\xi \in X; \xi s = \xi \text{ for all } s \in S\}$, and let Z be the closed vector subspace of X generated by $\{\xi - \xi s; s \in S, \xi \in X\}$. Then $X = Y \oplus Z$ and (E_α) converges strongly to the projection of X onto Y .*
- (3) *Let C_ξ be weakly compact for any $\xi \in X$. If $P: X \rightarrow Y$ is the projection onto Y associated with the direct sum decomposition of (2), then $P(\xi) = Y \cap C_\xi$ for all $\xi \in X$.*

Proof.

(1) Let M be a topologically right invariant mean on $M(S)^*$. If \mathcal{M} is the convex set of all means on $M(S)^*$, it is well known that \mathcal{M} is weak* compact and that $M_0(S)$ is weak* dense in \mathcal{M} (see [3,11] for details). Therefore there is a net (μ_α) in $M_0(S)$ such that $\mu_\alpha \rightarrow M$ in the weak* topology. A subnet $(\xi \mu_\beta)$ of $(\xi \mu_\alpha)$ converges in the weak topology to an element ζ in the compact subset $C_\xi = \overline{\{\xi \mu; \mu \in M_0(S)\}}$. For every $s \in S$ and $f \in X^*$, we have

$$\begin{aligned} \langle f, \zeta s \rangle &= \langle sf, \zeta \rangle = \lim_{\beta} \langle sf, \xi \mu_\beta \rangle = \lim_{\beta} \langle \mu_\beta(sf), \xi \rangle = \lim_{\beta} \langle (\mu_\beta * \delta_s)f, \xi \rangle \\ &= \lim_{\beta} \langle \mu_\beta(\delta_s f), \xi \rangle = \lim_{\beta} \langle \mu_\beta, (\delta_s f)\xi \rangle = \lim_{\beta} \langle \mu_\beta, \delta_s(f\xi) \rangle \\ &= \langle M, \delta_s(f\xi) \rangle = \langle M, f\xi \rangle = \lim_{\beta} \langle \mu_\beta, f\xi \rangle \\ &= \lim_{\beta} \langle \mu_\beta f, \xi \rangle = \lim_{\beta} \langle f, \xi \mu_\beta \rangle = \langle f, \zeta \rangle. \end{aligned}$$

So $\zeta s = \zeta$ for every $s \in S$, that is, ζ is a fixed point under the action of S . If $\varepsilon > 0$, we determine $\alpha_1, \dots, \alpha_m$ and $s_1, \dots, s_m \in S$ such that $\sum_{i=1}^m \alpha_i = 1$ and $\|\xi' - \zeta\| < \varepsilon$, where $\xi' = \sum_{i=1}^m \alpha_i \xi s_i$. We have

$$\begin{aligned} \|E_\alpha[\xi] - \zeta\| &= \|\xi \mu_\alpha - \zeta\| \leq \|\xi \mu_\alpha - \xi' \mu_\alpha\| + \|\xi' \mu_\alpha - \zeta\| \\ &= \|\xi \sum_{i=1}^m (\alpha_i \mu_\alpha - \alpha_i \delta_{s_i} * \mu_\alpha)\| + \|\xi' \mu_\alpha - \zeta \mu_\alpha\| \\ &\leq k \|\xi\| \sum_{i=1}^m \alpha_i \|\mu_\alpha - \delta_{s_i} * \mu_\alpha\| + k \|\xi' - \zeta\| \\ &< k \|\xi\| \sum_{i=1}^m \|\mu_\alpha - \delta_{s_i} * \mu_\alpha\| + k\varepsilon. \end{aligned}$$

Hence

$$\lim_{\alpha} \|E_\alpha[\xi] - \zeta\| = 0.$$

For every $\mu \in M_0(S)$, we have $E_\alpha[\xi \mu] = \xi \mu * \mu_\alpha$. Hence

$$\begin{aligned} \lim_{\alpha} \|E_\alpha[\xi \mu] - \zeta\| &= \lim_{\alpha} \|\xi \mu * \mu_\alpha - \xi \mu_\alpha\| \\ &\leq k \lim_{\alpha} \|\xi\| \|\mu * \mu_\alpha - \mu_\alpha\| = 0 \end{aligned}$$

and $\lim_{\alpha} \|E_\alpha[\zeta] - \zeta\| = 0$ for every $\zeta \in C_\xi$. Now if there exists another fixed point ζ' in C_ξ , then for every α we would have

$$E_\alpha[\zeta'] = \int \zeta' s d\mu_\alpha(s) = \zeta'$$

and

$$\zeta' = \lim_{\alpha} E_\alpha[\zeta'] = \zeta.$$

This shows that $(E_\alpha[\xi])$ converges to the unique fixed point ζ in C_ξ . Now if there exists another asymptotically $M_0(S)$ -invariant (v_β) in $M_0(S)$, it is easy to see that $\lim_{\beta} E_\beta[\xi] = \zeta$.

(2) Define an operator $P: X \rightarrow X$ by $P(\xi) = \lim_{\alpha} E_\alpha[\xi]$. Since X is a right S -module, it is easy to see that $\|P\| \leq k$. For every $\xi \in X$, $P(\xi) \in Y$ and $P(P(\xi)) = \lim_{\alpha} P(\xi) \mu_\alpha = P(\xi)$, $P^2 = P$. If $\xi \in Y$, then $P(\xi) = \lim_{\alpha} E_\alpha[\xi] = \xi \in Y$. We conclude that Y is the closed subspace $P(X)$ of X . Let $\xi \in X$ and $s \in S$. We have

$$\begin{aligned} \|P(\xi - \xi s)\| &= \|\lim_{\alpha} E_\alpha[\xi - \xi s]\| = \lim_{\alpha} \|\xi \mu_\alpha - \xi \delta_s * \mu_\alpha\| \\ &\leq k \|\xi\| \lim_{\alpha} \|\delta_s * \mu_\alpha - \mu_\alpha\| = 0. \end{aligned}$$

So $\xi - \xi \delta_s \in Y^\perp$. Conversely if $\zeta \in Y^\perp$, $P(\zeta) = 0$ is the unique fixed point in C_ζ and, for every $\varepsilon > 0$, there exist $\alpha_1, \dots, \alpha_m$ and s_1, \dots, s_m in S such that $\sum_{i=1}^m \alpha_i = 1$ and $\|\sum_{i=1}^m \alpha_i \zeta \delta_{s_i}\| < \varepsilon$. Hence $\|\zeta - \sum_{i=1}^m \alpha_i (\zeta - \zeta \delta_{s_i})\| = \|\sum_{i=1}^m \alpha_i \zeta \delta_{s_i}\| < \varepsilon$, such that $\zeta \in Z$.
(3) This is trivial. \square

COROLLARY 1.

Let $M(S)^*$ have a topologically right invariant mean. Suppose further that (μ_α) is an asymptotically $M_0(S)$ -invariant in $M_0(S)$. Suppose that S acts from the right on a reflexive Banach space X . Then $(E_\alpha[\xi])$ converges to an element ζ in X that is S -invariant.

Proof. For every $\xi \in X$, the set $C_\xi = \overline{\text{co}}\{\xi s; s \in S\}$ is a weak*-closed subset of the closed ball of radius $k\|\xi\|$ in $(X^*)^*$. This ball is weak* compact by Alaoglu's theorem and hence C_ξ is weak compact in the reflexive space X (p. 111 of [17]). The statement is an immediate consequence of Theorem 1. \square

We recall that, for a semigroup S , $M_a(S)$ is the set of all measures $\mu \in M(S)$ such that both mappings $s \mapsto |\mu| * \delta_s$ and $s \mapsto \delta_s * |\mu|$ from S into $M(S)$ are weakly continuous (see [1] and [5] for definition). A semigroup S is called a *foundation semigroup* if $\bigcup\{\text{supp } \mu; \mu \in M_a(S)\}$ is dense in S . It is known that $M_a(S)$ admits a positive approximate identity with norm 1 [5].

A functional f in $M_a(S)^*$ is said to be *weakly almost periodic* if the set $\{f\mu; \mu \in M_a(S), \|\mu\| \leq 1\}$ is relatively weakly compact. We denote by $\text{wap}(M_a(S))$ the closed subspace of $M_a(S)^*$ consisting of all the weakly almost periodic functionals in $M_a(S)^*$ (for more on weakly almost periodic functionals, the reader is referred to [2] and [12]).

COROLLARY 2.

Let S be a foundation, locally compact Hausdorff topological semigroup with identity. Let $M(S)^*$ have a topologically right invariant mean. Suppose (μ_α) is an asymptotically $M_0(S)$ -invariant in $M_0(S)$.

- (1) If $f \in \text{wap}(M_a(S))$, then $E_\alpha[f]$ converges to an element f' in $\text{wap}(M_a(S))$ that is S -invariant, that is, $f's = s$ for all $s \in S$; f' is independent of the choice of (μ_α) .
- (2) Let M be the topologically left invariant mean on $\text{wap}(M_a(S))$. Then $Y = \mathbb{C}1$ and

$$Z = \{f \in \text{wap}(M_a(S)); \langle M, f \rangle = 0\}.$$

Note that if (μ_α) is an asymptotically $M_0(S)$ -invariant in $M_0(S)$, then

$$\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$$

for all $\mu \in M_0(S)$. We choose $v \in M_a(S) \cap M_0(S)$ and let $v_\alpha = \mu_\alpha * v$, $\alpha \in I$. Clearly $\|\mu * v_\alpha - v_\alpha\| \rightarrow 0$ for all $\mu \in M_0(S)$. If M is a weak* cluster point of (v_α) , then M is a topologically left invariant mean on $\text{wap}(M_a(S))$.

Proof.

- (1) We apply Theorem 1 to the Banach space $X = \text{wap}(M_a(S))$. If $f \in \text{wap}(M_a(S))$ and $s \in S$, let

$$\langle fs, \mu \rangle = \langle f, \delta_s * \mu \rangle, \quad \mu \in M_a(S).$$

We check the continuity of the mapping $s \mapsto fs$ for any $f \in \text{wap}(M_a(S))$. Let $f \in \text{wap}(M_a(S))$ and (e_α) be a bounded approximate identity in $M_a(S)$. It is easy to see that $fe_\alpha \rightarrow f$ in the weak* topology. By the compactness of $\{f\mu; \mu \in M_a(S), \|\mu\| \leq 1\}$,

we can suppose that $fe_\alpha \rightarrow f$ in the weak topology, passing to a subnet, if necessary. Clearly $f \in M_a(S)^*M_a(S)$, since $M_a(S)^*M_a(S)$ is a Banach space and $fe_\alpha \in M_a(S)^*M_a(S)$ for all $\alpha \in I$. By Lemma 2.2 in [6], $s \mapsto fs$ is continuous. Obviously X is a right Banach S -module.

If $f \in \text{wap}(M_a(S))$, then $\{fs; s \in S\}$ is relatively weakly compact. So Theorem 1 asserts that, for every $f \in X$, $(E_\alpha[f])$ converges to an element f' in X that is fixed; $f's = f'$ and $\langle f', \delta_s * \mu \rangle = \langle f', \mu \rangle$ for every $s \in S$ and every $\mu \in M_a(S)$.

(2) By Theorem 1, Y is the set of all $f \in X$ such that $fs = f$ for all $s \in S$. If $f \in Y$, then $\langle f, \delta_s * \mu \rangle = \langle f, \mu \rangle$ for all $s \in S$ and $\mu \in M_a(S)$. Consequently, $f = \langle f, \delta_e \rangle 1$.

We know that Z admits a dense subset consisting of functionals of the form $g = \sum_1^n f_i - f_i s_i$, where $f_i \in \text{wap}(M_a(S))$ and $s_i \in S$. By the definition of M , we have $\langle M, g \rangle = 0$ and so $Z \subseteq \{f \in \text{wap}(M_a(S)); \langle M, f \rangle = 0\}$. For the reverse inclusion, let $f \in \text{wap}(M_a(S))$ and $\langle M, f \rangle = 0$. By Theorem 1, we can write $f = f_1 + f_2$ for some $f_1 \in Y$ and $f_2 \in Z$, and so

$$\begin{aligned} 0 &= \langle M, f \rangle = \langle M, f_1 \rangle + \langle M, f_2 \rangle \\ &= \langle M, f_1 \rangle + 0 = \langle M, f_1 \rangle, \end{aligned}$$

since $Z \subseteq \{f \in \text{wap}(M_a(S)); \langle M, f \rangle = 0\}$. Consequently $\langle M, f_1 \rangle = 0$. For each $\mu \in M_0(S)$, we have

$$\begin{aligned} 0 &= \langle M, f_1 \rangle = \langle M, f_1 \mu \rangle = \lim_\alpha \langle \mu_\alpha, f_1 \mu \rangle \\ &= \lim_\alpha \langle f_1, \mu * \mu_\alpha \rangle = \lim_\alpha \langle f_1 \mu * \mu_\alpha, \delta_e \rangle \\ &= \lim_\alpha \langle f_1 \mu, \delta_e \rangle = \langle f_1, \mu \rangle. \end{aligned}$$

Therefore $f_1 = 0$. It follows that $Z = \{f \in \text{wap}(M_a(S)); \langle M, f \rangle = 0\}$. \square

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